A SQUARE TRIGONOMETRY

BY

RICHARD LEE KISSEL

A SQUARE TRIGONOMETRY

A Research Paper

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the Graduate Council of

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In Partial Fulfillment

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by

Richard Lee Kissel

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To the Graduate Council:

I am submitting herewith a Research Paper written by Richard Lee Kissel entitled "A Square Trigonometry." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Arts in Education, with a major in mathematics.

George Brietenstein

Major Professor

Accepted for the Council:

Wayne E. Stamper

Dean of the Graduate School

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Chapter 1

Introduction

Definition of the System

John C. Biddle has defined several systems of square functions. This paper will be concerned with one of those systems.

Consider a square whose vertices lie on the co-ordinate axes and whose center is located at the origin. This square can be described by the equation $|x| + |y| = 1$. A radius vector $\overrightarrow{OR}$ is drawn in the same manner as the radius vector in the development of circular functions. The angle A is defined as the angle measured counterclockwise from the positive x-axis to the radius vector.[1]

The following six functions are defined in terms of the co-ordinates of the point (P) of intersection of the radius vector and the square:

$$\text{san } A = x + y$$

$$\text{cus } A = x - y$$

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[1]John C. Biddle, "The Square Function: An Abstract System for Trigonometry," <u>The Mathematics Teacher</u>, Vol. 60 (1967), pp. 121-123.

$$\text{tin } A = \text{san } A/\text{cus } A = \frac{x + y}{x - y}$$

$$\text{resan } A = 1/\text{san } A = \frac{1}{x + y}$$

$$\text{recus } A = 1/\text{cus } A = \frac{1}{x - y}$$

$$\text{retin } A = 1/\text{tin } A = \text{cus } A/\text{san } A = \frac{x - y}{x + y}$$

Statement of the Problem

The problem is that of developing some of the properties of this "square trigonometry." This will be done in a logical manner, if possible, similar to a basic development of the circular functions.

Purpose of the Study

It is realized that when most people think of trigonometry, they think only in terms of triangles or practical applications. Therefore, the primary purpose of this study is to demonstrate the logical development of some of the mathematical properties of a trigonometric system.

Limitations of the Study

This study was limited to a single abstract system. Results of this study are not to be taken as characteristic of mathematical systems in general.

Review of the Literature

There are few sources of material on this subject. Among the authors of such materials are: John C. Biddle,[2] Larry R. Griffey,[3] Thomas Mikula,[4] and Robert J. Thomas.[5]

This paper was developed from ideas in Larry R. Griffey's paper, "The Square Function" and is based upon a system described by John C. Biddle in his article, "The Square Function" in *The Mathematics Teacher*.

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[2] Biddle, op. cit.

[3] Larry R. Griffey, "The Square Function" (unpublished Master's paper, Austin Peay State University, 1967)

[4] Thomas Mikula, "The Trigonometry of the Square," *The Mathematics Teacher*, Vol. 60 (1967), pp. 354-357.

[5] Robert J. Thomas, "A New Introduction to the Ideas and Methods of Trigonometry," *The Mathematics Teacher*, Vol. 54, (1961), pp. 427-435.

Chapter 2

A Chronological History of Elementary Trigonometry

1750 B.C. Plimpton 322: This is a tablet written in Old Babylonian script giving values for the secant of angles from 31 to 45 degrees.[1]

1650 B.C. Amhes (Rhind) Papyrus: This is a mathematical text relating to the mensuration of pyramids.[2]

140 B.C. Hipparchus: He probably introduced into Greece the division of a circle into 360 degrees.[3] He developed a kind of spherical trigonometry. He worked out a table of chords, that is of double sines of half the angle, and so began the science of trigonometry.[4] He wrote twelve books on the computations of chords of angles.[5]

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[1]Howard Eves, An Introduction to the History of Mathematics (New York: Holt, Rinehart and Winston, 1953), pp. 34-36.

[2]David E. Smith, History of Mathematics, II (New York: Dover Publications, Inc., 1953), 600.

[3]Eves, op. cit., p. 152.

[4]Smith, op. cit., Vol. I, p. 202.

[5]Florian Cajori, A History of Mathematics (New York: The Macmillan Company, 1919), p. 604.

100 A.D. Menelaus: He wrote books on chords and spherical trigonometry.[6]

150 A.D. Claudius Ptolmey: He developed a table which gave the values of the sine of angles from 0 degree to 90 degrees in fifteen minute intervals.[7]

510 A.D. Aryabhata: He wrote a treatise containing functions of angles.[8]

920 A.D. Albaleginus (al-Battani): He developed the law of cosines for a spherical triangle.[9] He also gave a rule for finding the altitude of the sun.[10]

980 A.D. Abu'l-Wefa: He is celebrated for his improvements in trigonometry, his introduction of the tangent, and his computation of tables of sines and tangents for every ten minutes; it is also very likely that he is entitled to credit for the use of secants and cosecants.[11]

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[6]Cajori, op. cit., p. 606.

[7]Eves, op. cit., p. 153.

[8]Cajori, op. cit., p. 608.

[9]Eves, op. cit., p. 194.

[10]Cajori, op. cit., p. 608.

[11]Smith, op. cit., Vol. I, p. 980.

1130 A.D. Jabir ibn Aflah (Gerber): He contributed Gerber's Theorem for spherical triangles.[12]

1150 A.D. Bhaskara: He gave a method for constructing a table of sines for every degree.[13]

1150 A.D. Gherado Cremonense (Gherado of Cremona): He used the word sinus in reference to the half-chord. This was the first recorded use of our modern names for the trigonometric functions.[14]

1250 A.D. Nasir ed-din al-Tusi: He wrote the first work on plane and spherical trigonometry considered independently of astronomy. His work was so well done that, had his work been known, Europeans of the 15th century might have spared their labors.[15]

1290 A.D. Kou Shou-king: He began the study of spherical trigonometry in China.[16]

1330 A.D. Richard of Wallingford: He wrote extensively on trigonometry.[17]

1375 A.D. Simon Bredon: He was one of the earliest

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[12]Eves, op. cit., p. 195.

[13]Cajori, op. cit., p. 615.

[14]Smith, op. cit., Vol. I, p. 202.

[15]Cajori, op. cit., p. 108.

[16]Smith, op. cit., Vol. I, p. 272.

[17]Ibid., p. 235.

European scholars to pay much attention to trigonometry.[18]

1430 A.D. Johann von Grmunden: He wrote a treatise on trigonometry.[19]

1435 A.D. Ulugh Beg: He was a Persian astronomer who compiled remarkable tables of sines and tangents for one minute intervals correct to eight or more decimals.[20]

1460 A.D. Georg Von Peurbach: He compiled a table of sines.[21]

1464 A.D. Johann Muller: He wrote the first European work that may be said to have been devoted solely to trigonometry.[22]

1543 A.D. Nicholas Copernicus: He wrote a treatise on trigonometry to aid in the development of his theory of the universe.[23]

1550 A.D. Georg Joachim Rhaeticus: He spent twelve years with hired computers forming two remarkable and still useful trigonometric

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[18] Smith, op. cit., Vol. I, p. 237.

[19] Ibid., p. 258.

[20] Eves., op. cit., p. 191.

[21] Smith, op. cit., Vol. I, p. 259.

[22] Ibid., p. 260. [23] Ibid., p. 348.

tables. One was a ten-place table, of all six of the trigonometric functions for every ten seconds of arc; the other was a fifteen-place table of sines for every ten seconds of arc. He was the first to define the trigonometric functions as ratios of the sides of a right triangle.[24]

1579 A.D. Francois Viete (Vieta): He published the first book in western Europe which systematically developed methods for solving plane and spherical triangles with the aid of all six trigonometric functions.[25]

1594 A.D. Thomas Blundeville: He published a work containing tables of sines, tangents and secants. This was the first fairly complete treatment of trigonometry in England.[26]

1595 A.D. Bartholomaus Pitiscus: His Trigonometry was the first satisfactory textbook published on the subject and the first book to bear this title. He also edited and perfected the table of sines of Rhaeticus.[27]

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[24]Eves, op. cit., p. 226.

[25]Ibid., p. 223.

[26]Smith, op. cit., Vol. I, p. 324.

[27]Ibid., p. 331.

1613 A.D. Willebrord Snell van Roijen (Snell): He set forth the properties of the polar triangle in spherical trigonometry.[28]

1640 A.D. Nicolaus Mercator: He wrote a treatise on trigonometry.[29]

1685 A.D. John Caswell: He published a work on trigonometry.[30]

1690 A.D. John Wallis: He encouraged statements of formulas by equations instead of by proportions.[31]

1710 A.D. Sir Issac Newton: He gave sine x and arcsine x in infinite series form.[32]

1710 A.D. Thomas-Fantel de Lagny: The first to set forth in clear form the periodicity of the functions.[33]

1729 A.D. Daniel Bernoulli: The first to use suitable notation for inverse trigonometric functions; in 1729 he used AS. to represent arcsine.[34]

1730 A.D. Abraham de Moive: "In his discussion of trigonometry he gave the theorem which bears

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[28]Smith, op. cit., Vol. I, p. 423.

[29]Ibid., p. 434. [30]Ibid., p. 413.

[31]Smith, op. cit., Vol. II, p. 612.

[32]Ibid. [33]Ibid.

[34]Cajori, op. cit., p. 223.

his name, $(\cos x + i \sin x)^n = \cos nx + i \sin nx$."[35] This theorem has become the keystone of analytic trigonometry.[36]

1736 A.D. L. Euler: In 1736 he used At for arc-tangent.[37]

1759 A.D. Kastner: "The first writer to define the functions expressly as pure number."[38]

1770 A.D. Johann Heinrich Lambert: He wrote on hyperbolic trigonometry.[39]

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[35]Smith, op. cit., Vol. I, p. 450.

[36]Eves, op. cit., p. 351.

[37]Cajori, op. cit., p. 223.

[38]Smith, op. cit., Vol. II, p. 613.

[39]Smith, op. cit., Vol. I, p. 480.

Chapter 3

Identity Relationships Among the Functions

An identity is an equation that holds true for all permissible values of the quantities involved.[1] The object of this chapter is to reduce an expression involving sums or differences or two functions to an expression involving a single fraction or product. It is also desirable to reduce an expression involving products or quotients of two functions to a simpler expression.

Sums and Differences of the Functions

$$\text{san } A \pm \text{tin } A = \text{san } A \pm \frac{\text{san } A}{\text{cus } A}$$

$$= \frac{\text{san } A \cdot \text{cus } A \pm \text{san } A}{\text{cus } A}$$

$$= \frac{\text{san } A \cdot (\text{cus } A \pm 1)}{\text{cus } A}$$

$$= \text{tin } A \cdot (\text{cus } A \pm 1)$$

$$\text{san } A \pm \text{resan } A = \text{san } A \pm \frac{1}{\text{san } A} = \frac{\text{san}^2 A \pm 1}{\text{san } A}$$

$$\text{san } A \pm \text{recus } A = \text{san } A \pm \frac{1}{\text{cus } A} = \frac{\text{san } A \cdot \text{cus } A \pm 1}{\text{cus } A}$$

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[1] Elbridge P. Vance, Trigonometry (Massachusetts: Addison-Wesley Publishing Company, Inc., 1956), p. 87.

$$\text{san } A \pm \text{retin } A = \text{san } A \pm \frac{\text{cus } A}{\text{san } A} = \frac{\text{san}^2 A \pm \text{cus } A}{\text{san } A}$$

$$\text{cus } A \pm \text{tin } A = \text{cus } A \pm \frac{\text{san } A}{\text{cus } A} = \frac{\text{cus}^2 A \pm \text{san } A}{\text{cus } A}$$

$$\text{cus } A \pm \text{resan } A = \text{cus } A \pm \frac{1}{\text{san } A} = \frac{\text{san } A \cdot \text{cus } A \pm 1}{\text{san } A}$$

$$\text{cus } A \pm \text{recus } A = \text{cus } A \pm \frac{1}{\text{cus } A} = \frac{\text{cus}^2 A \pm 1}{\text{cus } A}$$

$$\text{cus } A \pm \text{retin } A = \text{cus } A \pm \frac{\text{cus } A}{\text{san } A}$$

$$= \frac{\text{san } A \cdot \text{cus } A \pm \text{cus } A}{\text{san } A}$$

$$= \frac{\text{cus } A \cdot (\text{san } A \pm 1)}{\text{san } A}$$

$$= \text{retin } A \cdot (\text{san } A \pm 1)$$

$$\text{tin } A \pm \text{resan } A = \frac{\text{san } A}{\text{cus } A} \pm \frac{1}{\text{san } A} = \frac{\text{san}^2 A \pm \text{cus } A}{\text{san } A \cdot \text{cus } A}$$

$$\text{tin } A \pm \text{recus } A = \frac{\text{san } A}{\text{cus } A} \pm \frac{1}{\text{cus } A} = \frac{\text{san } A \pm 1}{\text{cus } A}$$

$$\text{tin } A \pm \text{retin } A = \frac{\text{san } A}{\text{cus } A} \pm \frac{\text{cus } A}{\text{cus } A} = \frac{\text{san}^2 A \pm \text{cus}^2 A}{\text{san } A \cdot \text{cus } A}$$

$$\text{resan } A \pm \text{recus } A = \frac{1}{\text{san } A} \pm \frac{1}{\text{cus } A} = \frac{\text{cus } A \pm \text{san } A}{\text{san } A \cdot \text{cus } A}$$

$$\text{resan } A \pm \text{retin } A = \frac{1}{\text{san } A} \pm \frac{\text{cus } A}{\text{san } A} = \frac{1 \pm \text{cus } A}{\text{san } A}$$

$$\text{recus } A \pm \text{retin } A = \frac{1}{\text{cus } A} \pm \frac{\text{cus } A}{\text{san } A} = \frac{\text{san } A \pm \text{cus}^2 A}{\text{san } A \cdot \text{cus } A}$$

Products of the Functions

$$\text{san } A \cdot \text{tin } A = \text{san } A \cdot \frac{\text{san } A}{\text{cus } A} = \frac{\text{san}^2 A}{\text{cus } A}$$

$$\text{san } A \cdot \text{resan } A = \text{san } A \cdot \frac{1}{\text{san } A} = 1$$

$$\text{san } A \cdot \text{recus } A = \text{san } A \cdot \frac{1}{\text{cus } A} = \text{tin } A$$

$$\text{san } A \cdot \text{retin } A = \text{san } A \cdot \frac{\text{cus } A}{\text{san } A} = \text{cus } A$$

$$\text{cus } A \cdot \text{tin } A = \text{cus } A \cdot \frac{\text{san } A}{\text{cus } A} = \text{san } A$$

$$\text{cus } A \cdot \text{recus } A = \text{cus } A \cdot \frac{1}{\text{cus } A} = 1$$

$$\text{cus } A \cdot \text{resan } A = \text{cus } A \cdot \frac{1}{\text{san } A} = \text{retin } A$$

$$\text{cus } A \cdot \text{retin } A = \text{cus } A \cdot \frac{\text{cus } A}{\text{san } A} = \frac{\text{cus}^2 A}{\text{san } A}$$

$$\text{tin } A \cdot \text{resan } A = \frac{\text{san } A}{\text{cus } A} \cdot \frac{1}{\text{san } A} = \text{recus } A$$

$$\text{tin } A \cdot \text{retin } A = \text{tin } A \cdot \frac{1}{\text{tin } A} = 1$$

$$\text{tin } A \cdot \text{recus } A = \frac{\text{san } A}{\text{cus } A} \cdot \frac{1}{\text{cus } A} = \frac{\text{san } A}{\text{cus}^2 A}$$

$$\text{resan } A \cdot \text{recus } A = \frac{1}{\text{san } A} \cdot \frac{1}{\text{cus } A} = \frac{1}{\text{san } A \cdot \text{cus } A}$$

$$\text{resan } A \cdot \text{retin } A = \frac{1}{\text{san } A} \cdot \frac{\text{cus } A}{\text{san } A} = \frac{\text{cus } A}{\text{san}^2 A}$$

$$\text{recus } A \cdot \text{retin } A = \frac{1}{\text{cus } A} \cdot \frac{\text{cus } A}{\text{san } A} = \text{resan } A$$

Quotients of the Functions

$$\frac{\text{san } A}{\text{cus } A} = \text{tin } A$$

$$\frac{\text{san } A}{\text{tin } A} = \text{san } A \cdot \text{retin } A = \text{cus } A$$

$$\frac{\text{san } A}{\text{resan } A} = \text{san } A \cdot \text{san } A = \text{san}^2 A$$

$$\frac{\text{san } A}{\text{recus } A} = \text{san } A \cdot \text{cus } A$$

$$\frac{\text{san } A}{\text{retin } A} = \text{san } A \cdot \text{tin } A = \frac{\text{san}^2 A}{\text{cus } A}$$

$$\frac{\text{cus } A}{\text{san } A} = \text{retin } A$$

$$\frac{\text{cus } A}{\text{tin } A} = \text{cus } A \cdot \text{retin } A = \frac{\text{cus}^2 A}{\text{san } A}$$

$$\frac{\text{cus } A}{\text{recus } A} = \text{cus } A \cdot \text{cus } A = \text{cus}^2 A$$

$$\frac{\text{cus } A}{\text{resan } A} = \text{cus } A \cdot \text{san } A$$

$$\frac{\text{cus } A}{\text{retin } A} = \text{cus } A \cdot \text{tin } A = \text{san } A$$

$$\frac{\text{tin } A}{\text{san } A} = \text{tin } A \cdot \text{resan } A = \text{recus } A$$

$$\frac{\text{tin } A}{\text{cus } A} = \text{tin } A \cdot \text{recus } A = \frac{\text{san } A}{\text{cus}^2 A}$$

$$\frac{\text{tin } A}{\text{resan } A} = \text{tin } A \cdot \text{san } A = \frac{\text{san}^2 A}{\text{cus } A}$$

$$\frac{\text{tin } A}{\text{recus } A} = \text{tin } A \cdot \text{cus } A = \text{san } A$$

$$\frac{\text{tin } A}{\text{retin } A} = \text{tin } A \cdot \text{tin } A = \text{tin}^2 A$$

$$\frac{\text{resan } A}{\text{san } A} = \text{resan } A \cdot \text{resan } A = \text{resan}^2 A$$

$$\frac{\text{resan } A}{\text{cus } A} = \text{resan } A \cdot \text{recus } A = \frac{1}{\text{san } A \cdot \text{cus } A}$$

$$\frac{\text{resan } A}{\text{tin } A} = \text{resan } A \cdot \text{retin } A = \frac{\text{cus } A}{\text{san}^2 A}$$

$$\frac{\text{resan } A}{\text{recus } A} = \text{resan } A \cdot \text{cus } A = \text{retin } A$$

$$\frac{\text{resan } A}{\text{retin } A} = \text{resan } A \cdot \text{tin } A = \text{recus } A$$

$$\frac{\text{recus } A}{\text{san } A} = \text{recus } A \cdot \text{resan } A = \frac{1}{\text{san } A \cdot \text{cus } A}$$

$$\frac{\text{recus } A}{\text{cus } A} = \text{recus}^2 A$$

$$\frac{\text{recus } A}{\text{tin } A} = \text{recus } A \cdot \text{retin } A = \text{resan } A$$

$$\frac{\text{recus } A}{\text{resan } A} = \text{tin } A$$

$$\frac{\text{recus } A}{\text{retin } A} = \text{recus } A \cdot \text{tin } A = \frac{\text{san } A}{\text{cus}^2 A}$$

$$\frac{\text{retin } A}{\text{san } A} = \text{retin } A \cdot \text{resan } A = \frac{\text{cus } A}{\text{san}^2 A}$$

$$\frac{\text{retin } A}{\text{cus } A} = \text{retin } A \cdot \text{recus } A = \text{resan } A$$

$$\frac{\text{retin } A}{\text{tin } A} = \text{retin}^2 A$$

$$\frac{\text{retin } A}{\text{resan } A} = \text{retin } A \cdot \text{san } A = \text{cus } A$$

$$\frac{\text{retin } A}{\text{recus } A} = \text{retin } A \cdot \text{cus } A = \frac{\text{cus}^2 A}{\text{san } A}$$

Sums and Differences of Squares of the Functions

$$\text{san}^2 A \pm \text{tin}^2 A = \text{san}^2 A \pm \frac{\text{san}^2 A}{\text{cus}^2 A} = \frac{\text{san}^2 A \cdot (\text{cus}^2 A \pm 1)}{\text{cus}^2 A}$$

$$= \text{tin}^2 A \cdot (\text{cus}^2 A \pm 1)$$

$$\text{san}^2 A \pm \text{resan}^2 A = \text{san}^2 A \pm \frac{1}{\text{san}^2 A} = \frac{\text{san}^4 A \pm 1}{\text{san}^2 A}$$

$$\text{san}^2 A \pm \text{resan}^2 A = \text{san}^2 A \pm \frac{1}{\text{cus}^2 A} = \frac{\text{san}^2 A \cdot \text{cus}^2 A \pm 1}{\text{cus}^2 A}$$

$$\text{cus}^2 A \pm \text{resan}^2 A = \text{cus}^2 A \pm \frac{1}{\text{san}^2 A} = \frac{\text{cus}^2 A \cdot \text{san}^2 A \pm 1}{\text{san}^2 A}$$

$$\text{cus}^2 A \pm \text{recus}^2 A = \text{cus}^2 A \pm \frac{1}{\text{cus}^2 A} = \frac{\text{cus}^4 A \pm 1}{\text{cus}^2 A}$$

$$\text{cus}^2 A \pm \text{retin}^2 A = \text{cus}^2 A \pm \frac{\text{cus}^2 A}{\text{san}^2 A} = \frac{\text{cus}^2 A}{\text{san}^2 A} \cdot (\text{san}^2 A \pm 1)$$

$$= \text{retin}^2 A \cdot (\text{san}^2 A \pm 1)$$

$$\text{tin}^2 A \pm \text{recus}^2 A = \frac{\text{san}^2 A}{\text{cus}^2 A} \pm \frac{1}{\text{cus}^2 A} = \frac{\text{san}^2 A \pm 1}{\text{cus}^2 A}$$

$$\text{resan}^2 A \pm \text{retin}^2 A = \frac{1 \pm \text{cus}^2 A}{\text{san}^2 A}$$

Chapter 4

Numerical Properties of the Functions

Domain and Range

The domain of each function is the set of all real numbers excluding those for which the function is undefined. The set of all possible values for each function is its range.

Table 1

Domain and Range of the Functions

Function	Domain of Function	Range of Function
san A	R(all real numbers)	$-1 \leq$ san A ≤ 1
cus A	R	$-1 \leq$ cus A ≤ 1
tin A	R except($\pi/4 \pm n\pi$, $n \in z$)	R
resan A	R except($3\pi/4 \pm n\pi$)	resan A ≤ -1 resan A ≥ 1
recus A	R except($\pi/4 \pm n\pi$)	recus A ≤ -1 recus A ≥ 1
retin A	R except($3\pi/4 \pm n\pi$)	R

Signs and Values

It will be convenient to divide the rectangular co-ordinate system into eight equal regions in order to examine the behavior of the functions.

For A in the first region, $(0, \pi/4)$, $x + y > 0$ and $x - y > 0$; thus, all functions are positive in the first region. By continuing this method, the following table can be developed.

Table 2

Signs of the Functions in the Eight Regions

A in Region	san A	cus A	tin A	resan A	recus A	retin A
I	+	+	+	+	+	+
II	+	-	-	+	-	-
III	+	-	-	+	-	-
IV	-	-	+	-	-	+
V	-	-	+	-	-	+
VI	-	+	-	-	+	-
VII	-	+	-	-	+	-
VIII	+	+	+	+	+	+

Signs of the functions at 0, $\pi/4$, $\pi/2$, . . ., $7\pi/4$ have been omitted from the above table. The following table gives this information. These angles will be called the octantal angles.

Table 3

Signs of the Functions for Octantal Angles

A	san A	cus A	tin A	resan A	recus A	retin A
0	+	+	+	+	+	+
$\frac{\pi}{4}$	+	0	unde-fined	+	unde-fined	0
$\frac{\pi}{2}$	+	-	-	+	-	-
$\frac{3\pi}{4}$	0	-	0	unde-fined	-	unde-fined
π	-	-	+	-	-	+
$\frac{5\pi}{4}$	-	0	unde-fined	-	unde-fined	0
$\frac{3\pi}{2}$	-	+	-	-	+	-
$\frac{7\pi}{4}$	0	+	0	unde-fined	+	unde-fined

The next table will give the values of the functions in the octants and will include the end points of the regions.

Table 4

Values of the Functions

A	san A	cus A	tin A	resan A	recus A	retin A
0 to $\frac{\pi}{4}$	1	1 to 0	1 to $+\infty$	1	1 to $+\infty$	1 to 0
$\frac{\pi}{4}$ to $\frac{\pi}{2}$	1	0 to -1	$-\infty$ to -1	1	$-\infty$ to -1	0 to -1
$\frac{\pi}{2}$ to $\frac{3\pi}{4}$	1 to 0	-1	-1 to 0	1 to $+\infty$	-1	-1 to $-\infty$
$\frac{3\pi}{4}$ to π	0 to -1	-1	0 to 1	$-\infty$ to -1	-1	$+\infty$ to 1
π to $\frac{5\pi}{4}$	-1	-1 to 0	1 to $+\infty$	-1	-1 to $-\infty$	1 to 0
$\frac{5\pi}{4}$ to $\frac{3\pi}{2}$	-1	0 to 1	$-\infty$ to -1	-1	$+\infty$ to 1	0 to -1
$\frac{3\pi}{2}$ to $\frac{7\pi}{4}$	-1 to 0	1	-1 to 0	-1 to $-\infty$	1	-1 to $-\infty$
$\frac{7\pi}{4}$ to 2π	0 to 1	1	0 to 1	$+\infty$ to 1	1	$+\infty$ to 1

Reduction Formulas

It is convenient to be able to express a function of any angle in terms of some function of an angle between 0 and $\pi/4$.

The following relationships were formulated by observing the graphs and checking a table of values of the functions. They may be verified for any angle A by using the definition of the function and the definition of the system.

$$\text{cus } A = \text{san}(\pi/2 + A)$$

$$\text{retin } A = -\text{tin}(\pi/2 + A)$$

$$\text{recus } A = \text{resan}(\pi/2 + A)$$

$$\text{cus } A = -\text{cus}(\pi + A)$$

$$\text{san } A = -\text{san}(\pi + A)$$

$$\text{tin } A = \text{tin}(\pi + A)$$

$$\text{retin } A = \text{retin}(\pi + A)$$

$$\text{cus}(\pi/4 + A) = \text{retin}(\pi/4 + A) = -\text{cus}(\pi/4 - A)$$
$$= -\text{retin}(\pi/4 - A)$$

$$\text{tin}(\pi/4 + A) = \text{recus}(\pi/4 + A) = -\text{tin}(\pi/4 - A)$$
$$= -\text{recus}(\pi/4 - A)$$

$$\text{tin}(\pi/2 + A) = \text{san}(\pi/2 + A) = \text{cus}(\pi/2 - A)$$

$$\text{retin}(\pi/2 + A) = -\text{resan}(\pi/2 + A) = \text{tin}(\pi/2 - A)$$

$$\text{tin}(3\pi/4 + A) = -\text{san}(3\pi/4 + A) = \text{san}(3\pi/4 - A)$$
$$= -\text{tin}(3\pi/4 - A)$$

$$\text{retin}(3\pi/4 + A) = -\text{resan}(3\pi/4 + A)$$
$$= \text{resan}(3\pi/4 - A)$$
$$= -\text{retin}(3\pi/4 - A)$$

Chapter 5

Periodicity and Basic Graphs

"One of the characteristic properties that distinguish trigonometric functions from the functions of elementary algebra is 'periodicity.'"[1]

One definition of periodicity is now given.

> Let f(x) be a function having as its domain a set of real numbers denoted by D. Let k be a real number different from zero for which it is true that $x \pm k$ is in D whenever x is in D. The function f(x) is called periodic with period k if $f(x) = f(x + k)$ for every value of x in D.[2]

Observing Table 4, it is seen that the functions have the following values of k for their periods.

Table 5

Periods of the Functions

Function	san A	cus A	tin A	resan A	recus A	retin A
Period(k)	2π	2π	π	2π	2π	π

It is now simple to graph the functions using information from Table 1 through Table 5.

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[1] Henry Sharp, Jr., Elements of Plane Trigonometry (New Jersey: Prentice-Hall, Inc., 1958), p. 53.

[2] Ibid., p. 57.

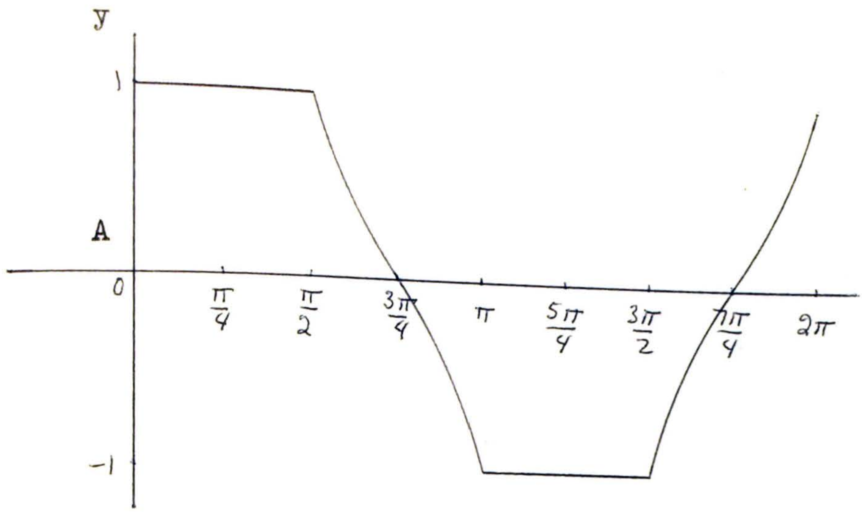

Figure 1

y = san A

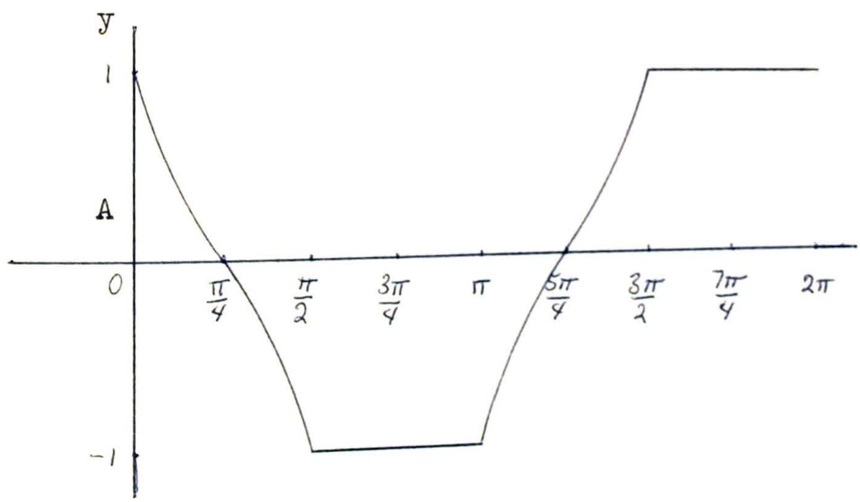

Figure 2

y = cus A

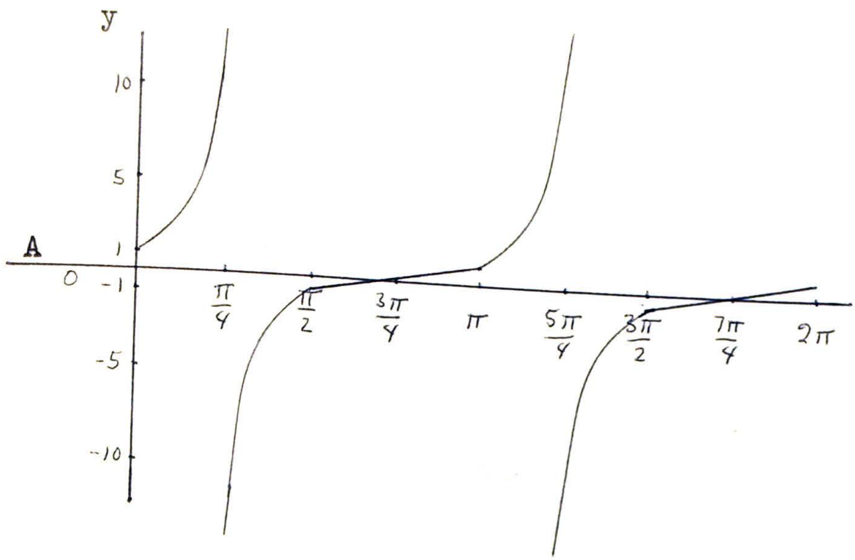

Figure 3

y = tin A

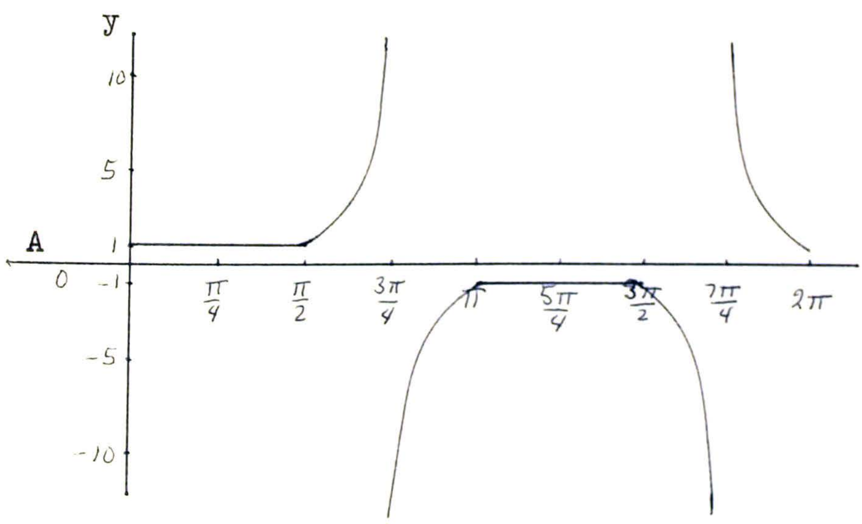

Figure 4

y = resan A

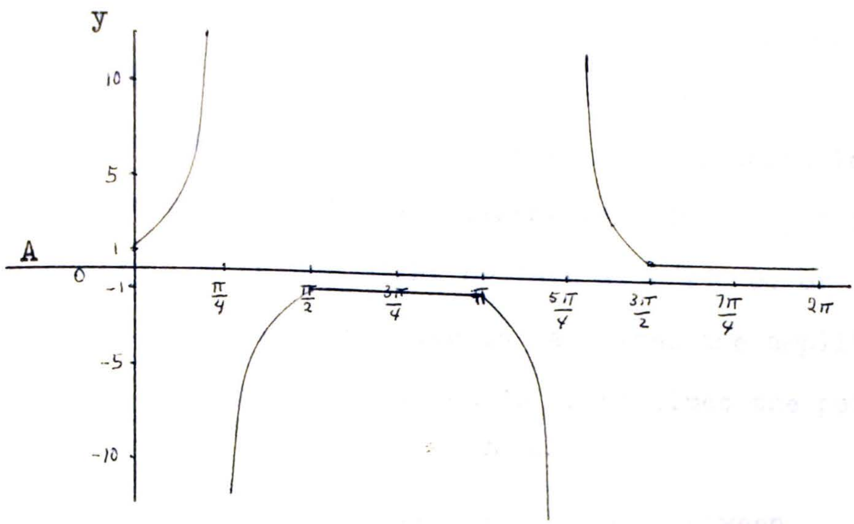

Figure 5

y = recus A

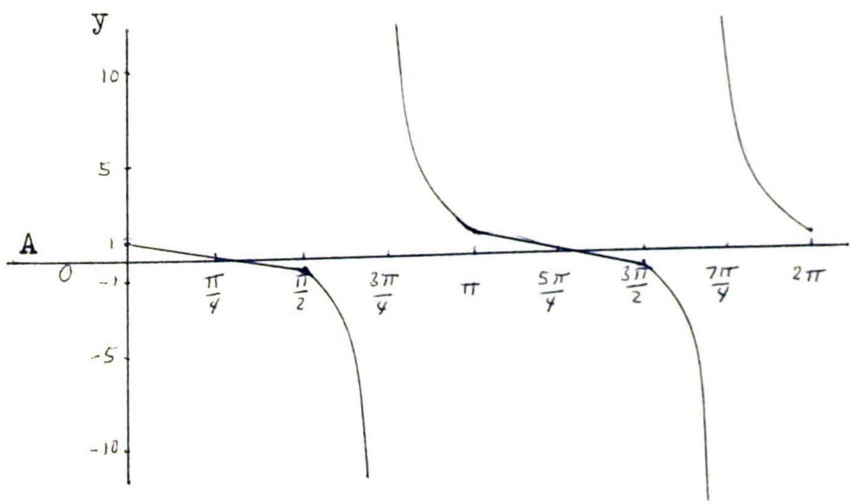

Figure 6

y = retin A

Mention should also be made of the graphs of $y = a \text{ san}(kA + b)$. The graph of $y = a \text{ san}(kA + b)$ differs from the graph of $y = \text{san } A$ in: its shift in the horizontal direction a distance $|b/k|$ (if $k > 0$ this displacement is to the right or left as $b/k > 0$, or $b/k < 0$), its amplitude which is $|a|$ times the amplitude of $y = \text{san } A$, its period which is $|1/k|$ times the period of $y = \text{san } A$.

Similar differences are noted between $y = a \text{ cus}(kA + b)$ and $y = \text{cus } A$; between $y = a \text{ tin}(kA + b)$ and $y = \text{tin } A$; and between the remaining functions and their general forms.

Chapter 6

Inverse Functions

In considering the relation y = san A, it may be desirable to talk about y, the san of A, but emphasis might also be given to the angle A, that is, A whose san is y. The latter relation shall be denoted by A = Arcsan y. It is convenient to denote the inverse relations to the other functions in a similar manner. That is, Arccus A represents the inverse cus of A, etc.

The following graphs are constructed using the information above concerning inverse relations.

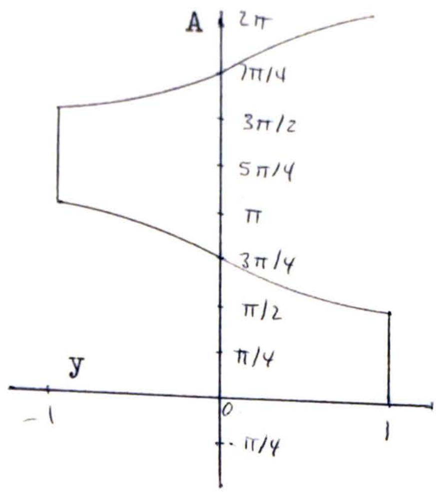

Figure 7

A = Arcsan y

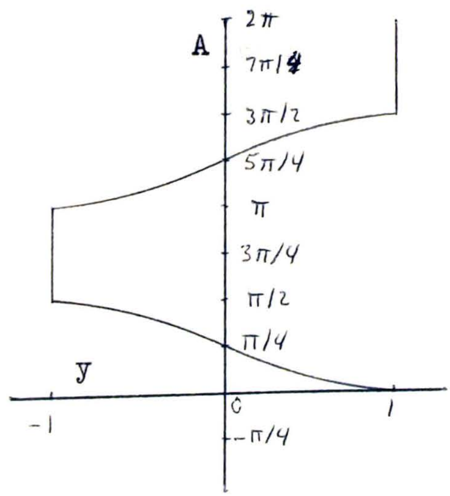

Figure 8

A = Arccus y

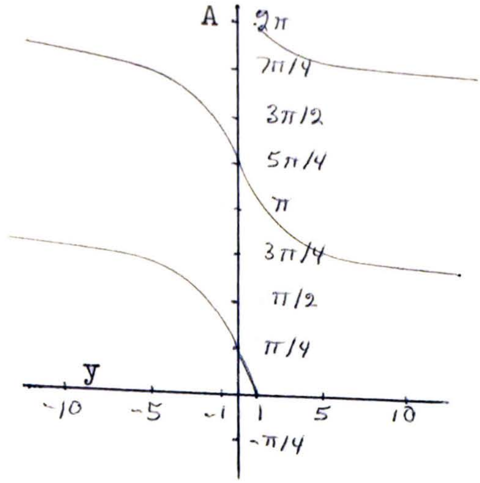

Figure 9

A = Arcretin y

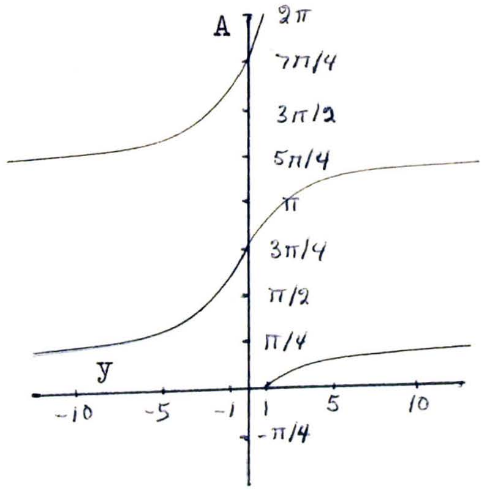

Figure 10

A = Arctin y

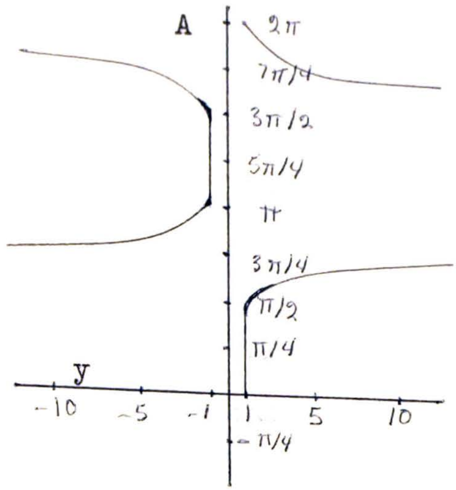

Figure 11

A = Arcresan y

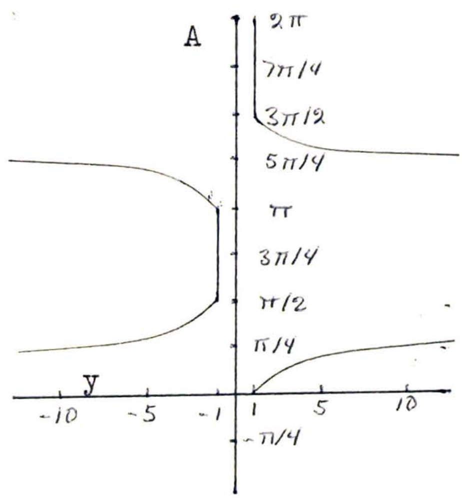

Figure 12

A = Arcrecus y

> A function of x is said to be single-valued if to one value of x there corresponds only one value of the function. A function of x is said to be multi-valued if two or more values of the function correspond to one value of x.[1]

It is obvious that the inverse relations are multi-valued. It is desirable to restrict these relations so that they are single-valued. This single value is called the principal value of the function. The notation for the principal value is the use of the capital A in writing the arc functions.

The range of the principal value of each of the inverse functions is as follows:

$\pi/2 \leq$ Arcsan A $\leq \pi$

$0 \leq$ Arccus A $\leq \pi/2$

$\pi/4 <$ Arctin A $< 5\pi/4$

$-\pi/4 <$ Arcretin A $< 3\pi/4$

$\pi/2 \leq$ Arcresan A $< 3\pi/4$ and $3\pi/4 <$ Arcresan A $\leq \pi$

$0 \leq$ Arcrecus A $< \pi/4$ and $\pi/4 <$ Arcrecus A $\leq \pi/2$

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[1]Vance, op. cit., p. 74.

Chapter 7

Double and Half-Angle Formulas

Double-Angle Formulas

It has been verified that for any angle A, $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$.

Let the co-ordinates of 2A be (x,y) and of A be (a,b). By direct substitution it is seen that $\frac{y}{x} = \frac{2(b/a)}{1-(b/a)^2}$ (for x, a $\neq$ 0) from which it follows that $y(a^2 - b^2) = 2abx$. The double and half-angle formulas are obtained from the latter equation.

Consider A in $(0, \pi/4)$. Obviously, $x + y = 1$, $a + b = 1$, and $x = 1 - y$. It follows that:

$$y(a^2 - b^2) = 2abx = 2ab(1 - y)$$

$$y(a^2 - b^2) + 2aby = 2ab$$

$$y\ (a^2 - b^2) + 2ab\ = 2ab$$

$$y = \frac{2ab}{a^2 - b^2 + 2ab}$$

Since $\text{san}2A - \text{cus}2A = 2y$, $\text{san}^2A - \text{cus}^2A = 4ab$, and $\text{san}\ A \cdot \text{cus}\ A = a^2 - b^2$, it follows that:

$$1/2(\text{san}2A - \text{cus}2A) = \frac{1/2(\text{san}^2A - \text{cus}^2A)}{\text{san } A\cdot\text{cus } A + 1/2(\text{san}^2A-\text{cus}^2A}$$

$$1/2(1 - \text{cus}2A) = \frac{1/2(1 - \text{cus}^2A)}{\text{cus } A + 1/2(1 - \text{cus}^2A}$$

$$1/2(1 - \text{cus}2A) = \frac{1 - \text{cus}^2A}{2\text{ cus } A + 1 - \text{cus}^2A}$$

$$1 - \text{cus}2A = \frac{2 - 2\text{cus}^2A}{2\text{ cus } A + 1 - \text{cus}^2A}$$

$$-\text{cus}2A = \frac{2 - 2\text{cus}^2A - 2\text{cus } A - 1 + \text{cus}^2A}{2\text{cus } A + 1 - \text{cus}^2A}$$

$$\text{cus}2A = \frac{2\text{ cus } A + \text{cus}^2A - 1}{2\text{ cus } A - \text{cus}^2A + 1}$$

The following formulas for A in the remaining seven octants are derived in a similar manner.

$$A \text{ in}(\pi/4,\ \pi/2)\ \ \text{san}2A = \frac{1 + 2\text{cus } A - \text{cus}^2A}{1 - 2\text{cus } A - \text{cus}^2A}$$

$$A \text{ in}(\pi/2,\ 3\pi/4)\ \ \text{cus}2A = \frac{\text{san}^2A + 2\text{san } A - 1}{\text{san}^2A - 2\text{san } A - 1}$$

$$A \text{ in}(3\pi/4,\ \pi)\ \ \text{san}2A = \frac{\text{san}^2A - 2\text{cus } A - 1}{1 - 2\text{san } A - \text{san}^2A}$$

$$A \text{ in}(\pi,\ 5\pi/4)\ \ \text{cus}2A = \frac{\text{cus}^2A - 2\text{cus } A - 1}{\text{cus}^2A + 2\text{cus } A - 1}$$

$$A \text{ in}(5\pi/4,\ 3\pi/2)\ \ \text{san}2A = \frac{\text{cus}^2A + 2\text{cus } A - 1}{\text{cus}^2A - 2\text{cus } A - 1}$$

$$A \text{ in}(3\pi/2,\ 7\pi/4)\ \ \text{cus}2A = \frac{\text{san}^2A - 2\text{san } A - 1}{1 - 2\text{san } A - \text{san}^2A}$$

$$A \text{ in}(7\pi/4,\ 2\pi)\ \ \text{san}2A = \frac{2\text{san } A + \text{san}^2A - 1}{2\text{san } A - \text{san}^2A + 1}$$

Half-Angle Formulas

The half-angle formulas are obtained from the double-angle formulas by solving for the function of A and then substituting 1/2 A for A in the resulting equation. This is the half-angle formula for A in $(0, \pi/4)$:

$$\text{cus } 1/2\ A = \frac{1 - \text{cus } A + \sqrt{2\text{cus}^2 A + 2}}{1 + \text{cus } A}$$

The remaining formulas are similarly derived.

Chapter 8

Applications of this System

The widespread realization by many laymen of the part played by mathematics and science in modern life presents educators with a challenge which must be answered. It is heartening to discover that so many intelligent people have become aware that mathematics is the lifeblood of modern science and also that without at least a rudimentary knowledge of the subject they are automatically debarred from a participation in and an understanding of many of the activities of this Space Age.

To maintain the enthusiasm which has been thus engendered, teachers must present mathematics in a manner which will arouse the interest of the student, and which will enable him from the outset to glimpse the value and fascination of the subject. The teacher must also offer the course in a way which will suggest that opportunities will be opened in later life if the student is equipped to take his place in an age when science and mathematics will play a greater part than in the past. All this applies with special force to the presentation of trigonometry.[1]

There are three uses for the mathematical system herein developed. It may be used briefly in a trigonometry course as an introduction to the basic ideas and methods. It may be used more extensively in a teacher training course to make the teachers-to-be think in the way that they should make their future students think. It may be used in an advanced class, even more extensively, to study the development of a mathematical system per se.[2]

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[1]Alice L. Griswold and Alfred Hooper, A Modern Course in Trigonometry (New York: Henry Holt & Co., 1959), p. v.

[2]Robert J. Thomas, "A New Introduction to the Ideas and Methods of Trigonometry," The Mathematics Teacher, Vol. 54, (1961), p. 427.

An application of this system would be to provide a breather to break the monotony in the usual trigonometry class. It may be used to provide enrichment for the better students or to provide a topic for discussion in the mathematics club.[3]

If this paper causes a single person to wish to learn more about mathematics, then it will have achieved its purpose.

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[3]Biddle, op. cit., p. 121.

BIBLIOGRAPHY

Biddle, John C. "The Square Function: An Abstract System for Trigonometry," The Mathematics Teacher, 60:121-123, February, 1967.

Cajori, Florian. A History of Mathematics. New York: Macmillan Co., 1919.

Eves, Howard. An Introduction to the History of Mathematics. New York: Holt, Rinehart & Winston, 1953.

Griffey, Larry R. "The Square Function." Unpublished Master's paper, Austin Peay State University, 1967.

Griswold, Alice L. and Alfred Hooper. A Modern Course in Trigonometry. New York: Henry Holt & co., 1959.

Mikula, Thomas. "The Trigonometry of the Square," The Mathematics Teacher, 60:427-435, April, 1967.

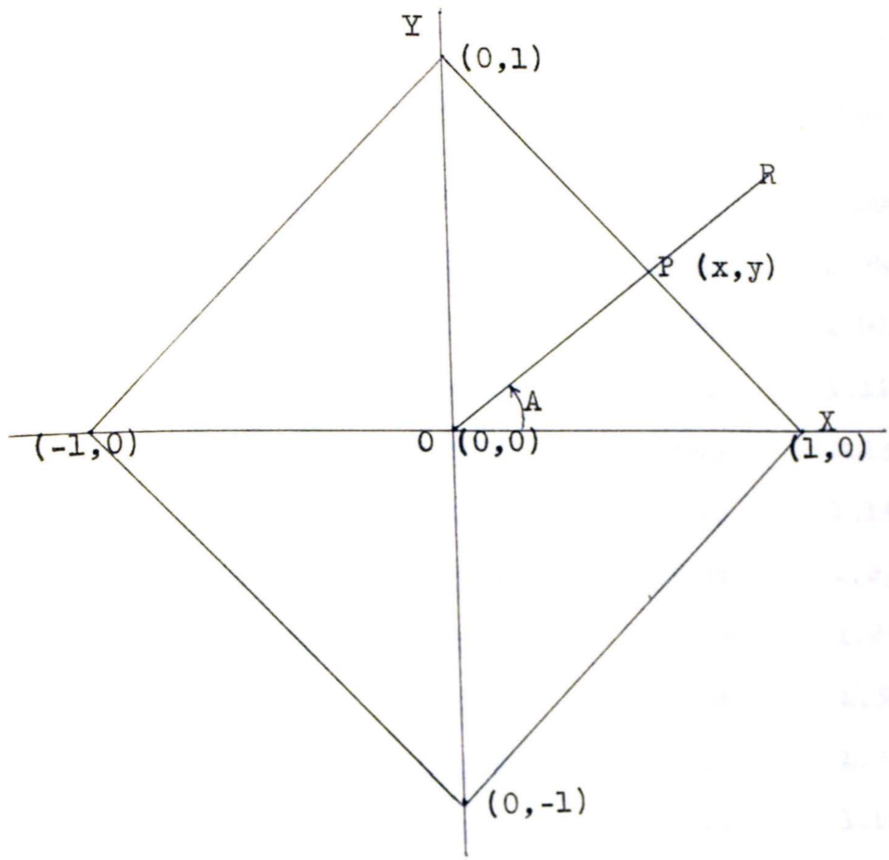
Sharp, Henry Jr. Elements of Plane Trigonometry. New Jersey: Prentice-Hall, 1958.

Smith, David E. History of Mathematics. 2 vols. New York: Dover Publications, 1953.

Thomas, Robert J. "A New Introduction to the Ideas and Methods of Trigonometry," The Mathematics Teacher, 54:427-435, October, 1961.

Vance, Elbridge P. Trigonometry. Massachusetts: Addison-Wesley Publishing Co., 1954.

APPENDIXES



A. Figure 13

Definition of the System

B. Values of the Functions

A in Degrees	A in Radians	san A	cus A	tin A
0	0.00000	1.0000	1.0000	1.0000
1	0.01745	1.0000	0.0657	1.0555
2	0.03491	1.0000	0.9325	1.0724
3	0.05236	1.0000	0.9004	1.1106
4	0.06981	1.0000	0.8693	1.1504
5	0.08727	1.0000	0.8391	1.1918
6	0.10472	1.0000	0.8098	1.2349
7	0.12217	1.0000	0.7813	1.2799
8	0.13963	1.0000	0.7536	1.3270
9	0.15708	1.0000	0.7265	1.3765
10	0.17453	1.0000	0.7002	1.4282
11	0.19199	1.0000	0.6745	1.4826
12	0.20944	1.0000	0.6494	1.5399
13	0.22689	1.0000	0.6249	1.6003
14	0.24435	1.0000	0.6009	1.6642
15	0.26180	1.0000	0.5773	1.7322
16	0.27925	1.0000	0.5593	1.8041
17	0.29671	1.0000	0.5317	1.8808
18	0.31416	1.0000	0.5095	1.9627
19	0.33161	1.0000	0.4877	2.0504
20	0.34907	1.0000	0.4663	2.1445
21	0.36652	1.0000	0.4452	2.2462
22	0.38397	1.0000	0.4245	2.3557

Values of the Functions (continued)

A in Degrees	A in Radians	san A	cus A	tin A
23	0.40143	1.0000	0.4040	2.4752
24	0.41888	1.0000	0.3839	2.6048
25	0.43633	1.0000	0.3640	2.7473
26	0.45379	1.0000	0.3443	2.9044
27	0.47124	1.0000	0.3249	3.0779
28	0.48869	1.0000	0.3057	3.2712
29	0.50615	1.0000	0.2867	3.4880
30	0.52360	1.0000	0.2679	3.7327
31	0.54105	1.0000	0.2493	4.0112
32	0.55851	1.0000	0.2309	4.3309
33	0.57596	1.0000	0.2126	4.7037
34	0.59341	1.0000	0.1944	5.1449
35	0.61807	1.0000	0.1763	5.6721
36	0.62832	1.0000	0.1584	6.3131
37	0.64577	1.0000	0.1405	7.1174
38	0.66323	1.0000	0.1228	8.1433
39	0.68068	1.0000	0.1051	9.5147
40	0.69813	1.0000	0.0875	11.4286
41	0.71559	1.0000	0.0699	14.3062
42	0.73304	1.0000	0.0524	19.0840
43	0.75049	1.0000	0.0349	28.6533
44	0.76795	1.0000	0.0175	57.1429
45	0.78540	1.0000	0.0000	∞

It is obvious that the values for resan A, recus A, and retin A are not given. By way of explanation; for A in $(0, \pi/4)$, resan A = 1, cus A = retin A, and tin A = recus A.