REVERSIBLE PRIME NUMBERS

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Roy Ronald Medlock

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To the Graduate Council:

I am submitting herewith a Research Paper written by Roy Ronald Medlock entitled "Reversible Prime Numbers". I recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Arts, with a major in Mathematics.

William G. Stokes
Major Professor

Accepted for the Council:

William H. Ellis
Dean of the Graduate School

REVERSIBLE PRIMES

The applicability of number theory to problems concerning the physical world is extremely rare; but the theory of numbers has, on the other hand, been a strong influence in the development of higher pure mathematics.

There are many branches of number theory. Among these is the study of prime numbers and the properties and characteristics they possess. The study of prime numbers, like all branches of number theory, was influenced primarily by man's insatiable curiousity--the drive to know and do everything.

The main purpose of this paper is to investigate a subset of the prime numbers, namely reversible primes, and to develop certain interesting ideas and characteristics of reversible prime numbers.

Among the positive integers there is a subclass of peculiar importance, the class of primes. A number P is said to be prime if:

(1) $P > 1$
(2) P has no positive divisors except 1 and P.[1]

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[1]G. H. Hardy and E. M. Wright, The Theory of Numbers. (Oxford at the Clarendon Press, 1962), P. 12.

Another definition is dependent on the fact that natural numbers may be classified as being either prime, composite, or 1, the number 1 falling into a class by itself, since it is neither prime nor composite. A natural number is prime if it has exactly two different factors, namely 1 and itself.[2]

The fundamental theorem of arithmetic states that any integer greater than 1 can be expressed as a product of primes, in one and essentially only one, way. Thus the prime numbers are building bricks from which all other integers may be made. Accordingly, the prime numbers have received much study, and considerable efforts have been spent trying to determine the nature of their distribution in the sequence of positive integers. The chief results obtained in antiquity are Euclid's proof of the infinitude of primes and Eratosthenes' sieve for finding all primes below a given integer n.[3]

Euclid's proof that there are an infinite number of primes has been universally regarded by mathematicians as a model of mathematical elegance. The proof employs the reductio ad absurdum, or indirect method, and runs as follows:

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[2]Melvin L. Keedy, Number Systems: A Modern Introduction. (Addison Wesley, Reading, Mass., 1965), P. 43.

[3]Howard Eves, An Introduction to the History of Mathematics. (Holt, Rinehard, and Winston, N. Y., 1962), P. 144-145.

Assume there is a finite number of primes; $P_1, P_2, P_3 \ldots P_n$. Let N be $(P_1 \cdot P_2 \cdot P_3 \ldots P_n) + 1$. Since N is greater than 1, it is either prime or composite. If N is prime we have a contradiction since it was assumed that $P_1, P_2, P_3 \ldots P_n$, made up the entire set of prime numbers. If N is composite it has a prime divisor P, but it is not divisible by $P_1 \ldots P_n$ since there will always be a remainder of one. Hence it must be divisible by a prime different from those listed. Thus our assumption that the set of prime numbers is finite has led to a contradiction and hence this assumption is false. Therefore the number of primes is infinite.[4]

Eratosthenes is noted for the following device, known as the sieve, for finding all the prime numbers less than a given number n. One writes down, in order and starting with 3, all the odd numbers less than n. The composite numbers in the sequence are then sifted out by crossing off, from 3, every third number, then from the next remaining number, 5,

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[4]*Ibid*., P. 118.

every fifth number, then from the next remaining number, 7, every seventh number, from the next remaining number 11, every eleventh number, and so on. In the process some numbers will be crossed off more than once. All the remaining numbers, along with the number 2, constitute the list of primes less than n.[5]

The following table for n = 50 will illustrate this method.

3	5	7	~~9~~	11	13	~~15~~	17
19	~~21~~	23	~~25~~	~~27~~	29	31	~~33~~
~~35~~	37	~~39~~	41	43	~~45~~	47	~~49~~

A considerable number of primes have the interesting property of reversibility. That is, certain primes yield prime numbers when the digits are reversed. For example, 37 is a reversible prime since its reverse, 73 is a prime. Let us first make some definitions relating to reversible primes.

Definition 1: Reversible Prime; Let A represent the digits of a prime number, where i ranges from 1 to n and $P = A_1 \; A_2 \; A_3 \; . \; . \; . \; A_{n-2} \; A_{n-1} \; A_n$. Then P is said to be a reversible prime if and only if the number $Q = A_n \; A_{n-1} \; A_{n-2} \; . \; . \; . \; A_1$ is also a prime number. We shall speak of a reversible prime and its reverse as a "pair" of reversible primes.

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[5] Oystein Ore, Number Theory and Its History, (McGraw-Hill Book Co., Inc., N. Y., 1948), pp. 64-67.

Note that the single digit primes 2, 3, 5, 7 trivially satisfy the definition of reversible primes.[6]

Definition 2: Order; The order of reversible primes is the number of digits of the prime. Hence, 37 is a reversible prime of order two; 143 is a reversible prime of order three; 1009 is a reversible prime of order four, etc.[7]

Definition 3: "Symmetrical" reversible primes; Let $P = A_1\ A_2\ A_3 \cdots A_{n-2}\ A_{n-1}\ A_n$ be a reversible prime of order n. Then P is said to be a symmetrical prime if and only if $A_1\ A_2 \cdots A_{n-1}\ A_n = A_n\ A_{n-1} \cdots A_2\ A_1$. The number 11 is the only symmetrical prime of order two and there are 15 symmetrical primes of order three. The smallest of these is 101 and the largest is 929.[8]

Definition 4: Symmetric twin primes; Two symmetric primes of order n, say P and Q ($P < Q$), are symmetric twins if and only if $Q - P = 10^{\frac{n-1}{2}}$. Consider for example the symmetric primes 10501 and 10601. Now, $10601 - 10501 = 100 = 10^2$ and $2 = \frac{5-1}{2}$. Thus these primes are symmetric twins.[9]

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[6]Niebaum, Jerome, "Bulletin of the Kansas Association of Teachers of Mathematics", April, 1967, P. 27.

[7]Ibid., P. 27.

[8]Ibid., P. 27.

[9]Ibid., P. 27.

We have shown that 11 and 101 are symmetric primes of order two and three and hence it would seem reasonable that there exist symmetrical primes of order four. For example, they might appear as 1331 or 3553 (neither of which is prime). Consider the form for a symmetrical prime of order four, say $P = abba$, where a and b represent digits of the prime. Then the number P has the value $1000 \cdot a + 100 \cdot b + 10 \cdot b + 1 \cdot a = 1001 \cdot a + 110 \cdot b = 11(91 \cdot a + 10 \cdot b)$ hence P has a factor 11 and therefore cannot be prime. Similarly it can be shown that all symmetric numbers with an even number of digits greater than 2 are not prime.[10] The proof is as follows:

Let $P = A_1 A_2 \ldots A_n$; where n is even and $n > 2$

Let $A_1 = A_n$

$A_2 = A_{n-1}$

$\vdots$

$A_{\frac{n}{2}} = A_{\frac{n}{2}+1}$

Then $P = 10^{n-1} \cdot A_1 + 10^{n-2} \cdot A_2 + \ldots 10^{n-n} \cdot A_n$

$$P = (10^{n-1} + 10^{n-n}) \cdot A_1 + (10^{n-2} + 10^{n-(n-1)}) \cdot A_2 + \ldots (10^{\frac{n}{2}} + 10^{\frac{n-2}{2}}) \cdot A_{\frac{n}{2}}$$

$$P = (10^{n-1} + 1)A_1 + (10^{n-2} + 10)A_2 \ldots + (10^{\frac{n}{2}} + 10^{\frac{n-2}{2}})A_{\frac{n}{2}}$$

[10] *Ibid.*, P. 28.

$$P = (10^{n-1} + 1)A_1 + 10(10^{n-3} + 1)A_2 \ldots + 10^{\frac{n-2}{2}} (10^1 + 10^0)A_{\frac{n}{2}}$$

$$P = (10^{n-1} + 1)A_1 + 10(10^{n-3} + 1)A_2 \ldots + 10^{\frac{n-2}{2}} (11)A_{\frac{n}{2}}$$

From the previous statement we see that $(10^{n-1} + 1)$; $(10^{n-3} + 1)$ . . . (11) are divisible by 11 because of the divisibility rule for 11. Therefore, P is divisible by 11 and hence not a prime number. Thus the only symmetric prime of an even number of digits is the number 11.

There are many somewhat unique sequences of symmetric primes. For example:

	131	is prime,
	10301	is prime,
	1003001	is prime,
and	100030001	is prime.

These four numbers have been verified as primes by tables and computers. It would seem that this sequence would continue, but not so. Computers have revealed that:

	10000300001	is divisible by 19,
	1000003000001	is divisible by 29,
	100000030000001	is divisible by 139,
	10000000300000001	is divisible by 61,[11]
and	1000000003000000001	is divisible by 59.

This is a good example to show the fallacy of inductive reasoning.

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[11] *Ibid.*, P. 29.

The following theorem says that in order for a prime to be reversible it must begin and end with 1, 3, 7, or 9. Let $A_1 A_2 \ldots A_n$ be a reversible prime of order n, $(n > 1)$ and let $D = \{1, 3, 7, 9\}$. Then $A_1 \in D$ and $A_n \in D$. The proof runs as follows:

Let $F = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Now $D \subset F$ and $D' \subset F$, where $D' = \{0, 2, 4, 5, 6, 8\}$. We know that $A_1 \in F$. Hence $A_1 \in D$ or $A_1 \in D'$. Assume $A_1 \in D'$. Then $A_n A_{n-1} \ldots A_2 A_1$ must end in 0, 2, 4, 5, 6, or 8, hence it is divisible by 2 or 5 and is not a reversible prime, contrary to hypothesis. Hence we conclude $A_1 \in D$. Similarly $A_n \in D$.[12]

It has been proved that sequences such as $\{4_{n-1}\}$, $n = \{1, 2, 3, \ldots\}$, contain an infinite number of primes. This is a special type of arithmetic sequence, $\{A_n + B\}$, which is an arithmetic progression, where $n = \{1, 2, 3, \ldots\}$, and A and B represent fixed numbers. When A and B are relatively prime, the sequence contains an infinite number of primes. This result is known as the theorem of Lejeune-Dirichlet.[13] The above leads to other questions. Consider $10^n + 1$, where $n = \{1, 2, 3, \ldots\}$, or 11, 101, 1001, 10001, 100001, etc. Does this contain an infinity of primes? If so, the

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[12] *Ibid.*, P. 29.

proof of the following conjectures concerning reversible primes would result.

(1) The set of symmetric primes is infinite.
(2) The set of reversible primes is infinite.

Note that since the set of symmetric primes is a proper subset of the set of reversible primes #2 follows from #1 as a corollary.

For symmetrical primes of order 3, 5, and 7, (refer to Appendix II), an increase in order yields an increasing sequence of symmetrical primes. Beginning with the digit 1, there are 5 symmetrical primes of order 3, 26 of order 5, and 189 of order 7. Beginning with the digit 3, there are 4 symmetrical primes of order 3, 24 of order 5, and 171 of order 7. Beginning with the digit 7, there are 4 symmetrical primes of order 3, 23 of order 5, and 155 of order 7. Beginning with the digit 9, there are 2 symmetrical primes of order 3, 18 of order 5, and 150 of order 7. In all there are 15 symmetrical primes of order 3, 96 of order 5, and 664 of order 7.

It seems that this trend of an increasing number of symmetrical primes as the order increases would continue. If this is true, then of course the set of symmetrical primes would be infinite and since the symmetrical primes are a subset of the set of reversible primes, it would follow that the set of reversible primes is infinite.

From a list of prime numbers (refer to Appendix I) it can be seen that the prime numbers in the 700's seem to be an extremely fertile source of reversible primes. The number 719 is not reversible since 917 is divisible by 7 and 773 is not reversible since 377 is divisible by 13. These are the only two out of the 14 primes in the 700's which are not reversible.

There are 4 reversible primes of order one, 8 of order two, 43 of order three, and 204 of order four. From a list of primes of order four, it is interesting to note that there appears a sequence of 10 consecutive primes, all of which are reversible primes: 1193, 1201, 1213, 1217, 1223, 1229, 1231, 1237, 1249, and 1259. Also of all the order four primes beginning with a 1, 3, 7, or 9 over 42% of them are reversible primes. There are 476 primes of order four and 204 of these are reversible primes. It can be seen that reversible primes are not as rare as might be expected. In fact, it seems that as the order of the prime numbers increases, so does the number of reversible primes.

One method of obtaining reversible primes would be to list all prime numbers of a certain order beginning with a 1, 3, 7, or 9 and check the reverse of each number against the list of primes. Of course, this would become very laborious for any order greater than three since the number of primes increases as the order increases. It would become so laborious, in fact, that the aid of a computer would be necessary.

Any set of primes of even order will have an even number of reversible primes since no even order set of numbers has a symmetrical prime and therefore each reversible prime of even order will give an entirely different prime number of the same order.

No attempt was made in this paper to exhaust the characteristics of reversible primes or conjectures which might be made concerning them. There are still many unanswered questions relative to reversible primes and in all probability some will remain unanswered.

The remarks in this paper verify the fact that the set of prime numbers, and especially reversible primes, offers a challenging and interesting study to anyone who is willing to devote the time and effort to this area. Perhaps such a study would fit into a high school mathematics program at some level, such as a part of a mathematics course or even as a topic for discussion in a mathematics club.

The study of reversible primes would likely arouse the curiosity of many students and perhaps lead to some important observations and developments in the field of prime numbers. At any rate, the person delving into recreational mathematics will find this topic both refreshing and rewarding.

APPENDIX I

Reversible Primes

Order One

2
3
5
7

Order Two

11	71
13	73
17	79
31	97
37	

Order Three

101	181	373	751	937
107	191	383	757	941
113	199	389	761	953
131	311	701	769	967
149	313	709	787	971
151	337	727	797	983
157	347	733	907	991
167	353	739	919	
179	359	743	929	

Order Four

1009	1283	1753	3203
1021	1301	1789	3221
1031	1321	1811	3251
1033	1381	1831	3257
1061	1399	1847	3271
1069	1409	1867	3299
1091	1429	1879	3301
1097	1439	1901	3319
1103	1453	1913	3343
1109	1471	1933	3347
1151	1487	1949	3359
1153	1499	1979	3371
1181	1511	3011	3373
1193	1523	3019	3389
1201	1559	3023	3391
1213	1583	3049	3407
1217	1597	3067	3433
1223	1601	3083	3463
1229	1619	3089	3467
1231	1657	3109	3469
1237	1669	3121	3511
1249	1723	3163	3527
1259	1733	3169	3541
1279	1741	3191	3571

Order Four Continued

3583	7321	7927	9479
3613	7349	7949	9491
3643	7433	7951	9497
3697	7457	7963	9521
3719	7459	9001	9533
3733	7481	9011	9547
3767	7507	9013	9551
3803	7523	9029	9601
3821	7529	9041	9613
3851	7547	9103	9643
3853	7561	9127	9661
3889	7577	9133	9679
3911	7589	9161	9721
3917	7603	9173	9749
3929	7643	9209	9769
7027	7649	9221	9781
7043	7673	9227	9787
7057	7681	9241	9791
7121	7687	9257	9803
7177	7699	9293	9833
7187	7717	9341	9857
7193	7757	9349	9871
7207	7817	9403	9883
7219	7841	9421	9923
7229	7867	9437	9931
7253	7879	9439	9941
7297	7901	9467	9967

APPENDIX II

Symmetrical Primes Order Three

101	313	727	919
131	353	757	929
151	373	787	
181	383	797	
191			

Symmetrical Primes Order Five

10201	30103	70207	90709
10301	30203	70507	91019
10501	30403	70607	93139
10601	30703	71317	93239
11311	30803	71917	93739
11411	31013	72227	94049
12421	31513	72727	94249
12721	32323	73037	94349
12821	32423	73237	94649
13331	33533	73637	94849
13831	34543	74047	94949
13931	34843	74747	95959
14341	35053	75557	96269
14741	35153	76367	96469
15451	35353	76667	96769
15551	35753	77377	97379
16061	36263	77477	97579
16361	36563	77977	97879
16561	36863	78487	98389
16661	37273	78787	98689
17471	37573	78887	
17971	38083	79397	
18181	38183	79697	
18481	38783	79997	
19391	39293		
19891			
19991			

Symmetrical Primes Order Seven

1003001
1008001
1022201
1028201
1035301
1043401
1055501
1062601
1065601
1074701
1082801
1085801
1092901
1093901
1114111
1117111
1120211
1123211
1126211
1129211
1134311
1145411
1150511
1153511
1160611
1163611
1175711
1177711
1178711
1180811
1183811
1186811
1190911
1193911
1196911
1201021
1208021
1212121
1215121
1218121
1221221
1235321
1242421
1243421
1245421
1250521
1253521
1257521
1262621
1268621
1273721
1276721
1278721
1280821
1281821
1286821
1287821
1300031
1303031
1311131
1317131
1327231
1328231
1333331
1335331
1338331
1343431
1360631
1362631
1371731
1374731
1390931
1407041
1409041
1411141
1412141
1422241
1437341
1444441
1447441
1452541
1456541
1461641
1463641
1464641
1469641
1486841
1489841
1490941
1496941
1508051
1513151
1520251
1532351
1535351
1542451
1548451
1550551
1551551
1556551
1557551
1565651
1572751
1579751
1580851
1583851
1589851
1594951
1597951
1598951
1600061
1609061
1611161
1616161
1628261
1630361
1633361
1640461
1643461
1646461
1654561
1657561
1658561
1660661
1670761
1684861
1685861
1688861
1695961
1703071
1707071
1712171
1714171
1730371
1734371
1737371
1748471
1755571
1761671
1764671
1777771
1793971
1802081
1805081
1820281
1823281
1824281
1826281
1829281
1831381
1832381
1842481
1851581
1853581
1856581
1865681
1876781
1878781
1879781
1880881
1881881
1883881
1884881
1895981
1903091
1908091
1909091
1917191
1924291
1930391
1936391
1941491
1951591
1952951
1957591
1958591
1963691
1968691
1969691
1970791
1976791
1981891
1982891
1984891
1987891
1988891
1993991
1995991
1998991
3001003
3002003
3016103
3026203
3064603
3065603
3072703
3073703
3075703
3083803
3089803
3091903
3095903
3103013
3106013
3127213
3135313
3140413
3155513

3158513	3417143	3732373	7035307
3160613	3424243	3743473	7036307
3166613	3425243	3746473	7041407
3181813	3427243	3762673	7046407
3187813	3439343	3763673	7057507
3193913	3441443	3765673	7065607
3196913	3443443	3768673	7069607
3198913	3444443	3769673	7073707
3211123	3447443	3773773	7079707
3212123	3449443	3774773	7082807
3218123	3452543	3781873	7084807
3222223	3460643	3784873	7087807
3223223	3466643	3792973	7093907
3228223	3470743	3793973	7096907
3233323	3479743	3799973	7100017
3236323	3485843	3804083	7114117
3241423	3487843	3806083	7115117
3245423	3503053	3812183	7118117
3252523	3515153	3814183	7129217
2356523	3517153	3826283	7134317
3258523	3528253	3829283	7136317
3260623	3541453	3836383	7141417
3267623	3553553	3842483	7145417
3272723	3558553	3853583	7155517
3283823	3563653	3858583	7156517
3285823	3569653	3863683	7158517
3286823	3586853	3864683	7159517
3288823	3589853	3867683	7177717
3291923	3590953	3869683	7190917
3293923	3591953	3871783	7194917
3304033	3594953	3878783	7215127
3305033	3601063	3893983	7226227
3307033	3607063	3899983	7246427
3310133	3618163	3913193	7249427
3315133	3621263	3916193	7250527
3319133	3627263	3918193	7256527
3321233	3635363	3924293	7257527
3329233	3643463	3927293	7261627
3331333	3646463	3931393	7267627
3337333	3670763	3938393	7276727
3343433	3673763	3942493	7278727
3353533	3680863	3946493	7291927
3362633	3689863	3948493	7300037
3364633	3698963	3964693	7302037
3365633	3708073	3970793	7310137
3368633	3709073	3983893	7314137
3380833	3716173	3991993	7324237
3391933	3717173	3994993	7327237
3392933	3721273	3997993	7347437
3400043	3722273	3998993	7352537
3411143	3728273	7014107	7354537

7362637	7690967	7977797	9320239
7365637	7693967	7984897	9324239
7381837	7696967	7985897	9329239
7388837	7715177	7987897	9332339
7392937	7718177	7996997	9338339
7401047	7722277	9002009	9351539
7403047	7729277	9015109	9357539
7409047	7733377	9024209	9375739
7415147	7742477	9037309	9384839
7434347	7747477	9042409	9397939
7436347	7750577	9043409	9400049
7439347	7758577	9045409	9414149
7452547	7764677	9046409	9419149
7461647	7772777	9049409	9433349
7466647	7774777	9067609	9439340
7472747	7778777	9073709	9440449
7475747	7782877	9076709	9446449
7485847	7783877	9078709	9451549
7486847	7791977	9091909	9470749
7489847	7794977	9095909	9477749
7493947	7807087	9103019	9492949
7507057	7819187	9109019	9493949
7508057	7820287	9110119	9495949
7518157	7821287	9127219	9504059
7519157	7831387	9128219	9514159
7521257	7832387	9136319	9526259
7527257	7838387	9169619	9529259
7540457	7843487	9173719	9547459
7562657	7850587	9174719	9556559
7564657	7856587	9179719	9558559
7576757	7865687	9185819	9561659
7586857	7867687	9196919	9577759
7592957	7868687	9199919	9583859
7594957	7873787	9200029	9585859
7600067	7884887	9209029	9586859
7611167	7891987	9212129	9601069
7619167	7897987	9217129	9602069
7622267	7913197	9222229	9604069
7630367	7916197	9223229	9610169
7632367	7930397	9230329	9620269
7644467	7933397	9231329	9624269
7654567	7935397	9255529	9626269
7662667	7938397	9269629	9632369
7665667	7941497	9271729	9634369
7666667	7943497	9277729	9645469
7668667	7949497	9280829	9650569
7669667	7957597	9286829	9657569
7674767	7958597	9289829	9670769
7681867	7960697	9318139	9686869

9700079
9709079
9711179
9714179
9724279
9727279
9732379
9733379
9743479
9749479
9752579
9754579
9758579
9762679
9770779
9776779
9779779
9781879
9782879
9787879
9788879
9795979
9801089
9807089
9809089
9817189
9818189
9820289
9822289
9836389
9837389
9845489
9852589
9871789
9888889
9889889
9896989
9902099
9907099
9908099
9916199
9918199
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